Journal of Statistical Physics, Vol. 45, Nos. 1/2, 1986

The Spherical-Model Limit in a Random Field

W. K. Theumann[1] **and José F. Fontanari**[2]

Received January 16, 1986; revision received May 2, 1986

The spherical-model limit $n \to \infty$ of the n-vector model in a random field, with either a statistically independent distribution or with long-range correlated random fields, is studied to demonstrate the correctness of the replica method in which the $n \to \infty$ and replica limits limits are interchanged, provided the replica and thermodynamic limits are taken in the right order, in the case of long-range correlated random fields. A scaling form for the two-point correlation function relevant to the first-order phase transition below the lower critical dimensionality of the random system is also obtained.

KEY WORDS: n-Vector model; long-range correlated random fields; replica method; interchange of replica and spherical-model limits; correlation functions for first-order phase transition.

1. INTRODUCTION

There is great current interest in systems with continuous or discrete symmetries in a quenched random field (see Ref. 1 for a recent review), and the exactly solvable spherical (or mean spherical) model and the equivalent spherical model limit ($n \to \infty$) of the n-vector model in a random field have received considerable attention in recent works,[2-7] following the original work of Lacour-Gayet and Toulouse[8] on the similar ideal Bose–Einstein condensation at constant volume.

Being exactly solvable, these models should enable a direct test on statistical procedures that average over a random field, in particular the often claimed to be unreliable but popular replica trick (See Ref. 9 for the

[1] Instituto de Física, Universidade Federal do Rio Grande do Sul, 90049 Porto Alegre, RS, Brazil.

[2] Instituto de Física e Química de São Carlos, Universidade de São Paulo, 13560 São Carlos, SP, Brazil.

0022-4715/86/1000-0099$05.00/0

application of the replica trick to physical systems). It has only recently been shown by Perez *et al.*[6] that a form of the replica method discussed by van Hemmen and Palmer[10] yields the correct free energy for the mean spherical model with a Gaussian distribution of statistically independent random fields.

The explicit calculation of thermodynamic properties by means of the replica trick demands, in general, an interchange of the thermodynamic and replica limits. The precise mathematical conditions that have to be fulfilled in order to justify this interchange and to demonstrate the existence of the replica limit are discussed in the work by van Hemmen and Palmer.[10] However, neither of these points has been proved in general.

The replica method has been used recently by Hornreich and Schuster[4] in the $n \to \infty$ limit for the n-vector model[11–13] with a Gaussian distribution of statistically independent random fields. Their explicit results are obtained through an interchange of the $n \to \infty$ and replica limits.

The purpose of the present paper is, first, to demonstrate that the interchange of the $n \to \infty$ and replica limits is correct. We show this by extending first the work of Hornreich and Schuster to: (i) statistically independent non-Gaussian distributions of random fields and (ii) long-range Gaussian correlated random fields that have been used recently for finite n by Kardar *et al.*[14] and Chang and Abrahams.[15] We contrast the results of the replica method with a direct exact calculation that avoids the replicas, and we can do this even for long-range Gaussian correlated random fields. Our results for the critical behavior obtained in this case can be checked against independent work by Carra and Chalker[7] in the $n \to \infty$ limit of the $1/n$ expansion of Ma,[17] and they do agree.

The free energy per particle and degree of freedom $f(\beta)$, in which $\beta = 1/kT$, for the $n \to \infty$ limit of the n-vector model with N particles follows as

$$-\beta f(\beta) = \lim_{N,n \to \infty} (Nn)^{-1} [\ln Q_N^{(n)}]_{\mathrm{av}} \tag{1}$$

where $[\cdots]_{\mathrm{av}}$ denotes the average over the random field distribution. We show here that if the $n \to \infty$ limit is taken before the thermodynamic limit, the free energy that is obtained is precisely that of the spherical model. Presumably, one may also interchange the limits or take them together, as in the case of the nonrandom model,[12] but we do not deal with this issue here.

Making use of the replica method, we write[10]

$$[\ln Q_N^{(n)}]_{\mathrm{av}} = \frac{d}{dm} \ln [Q_N^{(n,m)}]_{\mathrm{av}} \Big|_{m=0} \tag{2}$$

which is equivalent to the usual $m \to 0$ limit of $(Q_N^{(n,m)} - 1)/m$, where $Q_N^{(n,m)}$ is the m-times replicated partition function, with positive integer m. We assume, as usual, extension to real m and differentiation at $m = 0$, here and in the following.

Since one cannot calculate explicitly the right-hand side of Eq. (2) for finite N *and* n, one has to resort to an interchange of limits. Sticking to our assumption that the spherical-model limit $n \to \infty$ is taken before the $N \to \infty$ limit, we interchange the former with the replica limit and write

$$-\beta f(\beta) = \lim_{N \to \infty} \left[\frac{d}{dm} \phi_N^{(\mathrm{SM})}(m) \bigg|_{m=0} \right] \tag{3}$$

where

$$\phi_N^{(\mathrm{SM})}(m) \equiv N^{-1} \lim_{n \to \infty} n^{-1} \ln [Q_N^{(n,m)}]_{\mathrm{av}} \tag{4}$$

This is the procedure used by Hornreich and Schuster,[4] and to justify the steps involved in it, we resort to a replica-independent calculation.

We are also concerned with the interchange of thermodynamic and replica limits, which yields

$$-\beta f(\beta) = \frac{d}{dm} \phi^{(\mathrm{SM})}(m) \bigg|_{m=0} \tag{5}$$

where

$$\phi^{(\mathrm{SM})}(m) = \lim_{N \to \infty} \phi_N^{(\mathrm{SM})}(m) \tag{6}$$

A second purpose of this paper is to determine the scaling form of the structure factor, in the case of long-range correlated random fields, for the first-order phase transition discussed recently by Aharony and Pytte.[16] This is an interesting transition that occurs when the variance Δ of the random field distribution tends to zero and appears when $T < T_c^0$ and $d_c^0 < d < d_c$. Here T_c^0 is the critical temperature of the pure system, and d_c^0 and d_c are the lower critical dimensionalities of the pure and random systems, respectively.

The paper is organized as follows. In Section 2 we consider the spherical-model limit in the replica method, and in Section 3 we show that the same results are obtained in the replica-free calculation. This is done only briefly, since the steps one has to follow may be found in the more general although abstract work by Pastur.[2] We present our results on correlation functions in Section 4 and a summary with concluding remarks in Section 5.

2. REPLICA METHOD IN THE SPHERICAL-MODEL LIMIT

We follow Hornreich and Schuster[4] and consider the n-vector model of classical spins $\mathbf{S}_i = \{\sigma_1(i),..., \sigma_n(i)\}$ on the sites i ($i = 1,..., N$) of a regular d-dimensional lattice with $|\mathbf{S}_i| = n^{1/2}$, for all i, given by the Hamiltonian

$$H^{(n)} = -\frac{1}{2}\sum_{ij} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j - \sum_i \mathbf{H}_i \cdot \mathbf{S}_i - n^{1/2}\mathbf{H}_0 \cdot \sum_i \mathbf{S}_i \tag{7}$$

with ferromagnetic, translation-invariant, but not necessarily short-ranged interaction $J_{ij} = Jv_{ij}$ between spins at sites i and j in a uniform external field $n^{1/2}\mathbf{H}_0$ along the $[1,..., 1]$ direction and a quenched random field $\mathbf{H}_i$, with components H_{iv} ($v = 1,..., n$). In addition to the Gaussian distribution of statistically independent random fields considered in Ref. 4, we take a statistically independent non-Gaussian distribution with probability density

$$p(H_{iv}) = \frac{1}{2}[\delta(H_{iv} - h_0) + \delta(H_{iv} + h_0)] \tag{8}$$

for all i and v, and *Gaussian* random fields with components

$$H_v(\mathbf{k}) = N^{-1/2} \sum_{\mathbf{r}i} H_{iv} \exp(-i\mathbf{k} \cdot \mathbf{r}_i) \tag{9}$$

where $\mathbf{k}$ is a vector on reciprocal lattice space and $\mathbf{r}_i$ is the direct-lattice vector to site i, and

$$[H_v(\mathbf{k})]_{\text{av}} = 0 \tag{10a}$$

$$[H_\mu(\mathbf{k})\, H_v(\mathbf{k}')]_{\text{av}} = \Delta\delta_{\mu v}\, \delta(\mathbf{k} + \mathbf{k}')\, L_\theta(k) \tag{10b}$$

for all v. Here, $L_\theta(k) = k^\theta$ for any small $k = |\mathbf{k}|$ when $\theta \geqslant 0$ corresponding to uncorrelated fields ($\theta = 0$) or short-range oscillatory correlations ($\theta > 0$) in direct lattice space. The discrete set of values of $L_\theta(k)$ in Fourier lattice space for $\theta < 0$, corresponding to *long-range* correlated random fields in direct lattice space, is chosen to reproduce in the thermodynamic limit the continuum momentum space behavior of $L_\theta(k) \to k^\theta$ in the $k \to 0$ limit used by other recent authors[14,15] for the finite-n, n-vector model. As will be seen below, when the calculations are done with the replica method on a lattice, the value of $L_\theta(k)$ at $k = 0$ has to be treated properly. Since the critical behavior is determined by the continuum spectrum of $L_\theta(k)$ for small k that follows in the thermodynamic limit, with $k \propto N^{-1}$, as long as the number of sites remains finite, the discrete spectrum of $L_\theta(k)$ does not lead to a divergent $L_\theta(k)$ at $k = 0$ and one may choose an appropriate $L_\theta(0)$. The precise choice is irrelevant, as will be seen below, and we take $L_\theta(0) = \text{const} < \infty$.

We introduce next a set of replicated spin components, $\{\sigma_\nu^\alpha(i)\}$, $\alpha = 1,\ldots, m$, for each i and ν. Although one starts with noninteracting replicas, the calculation of the random-field average that defines the effective Hamiltonian

$$\mathscr{H}^{(n,m)} = -\beta^{-1} \ln[e^{-\beta H^{(n)}}]_{\text{av}} \tag{11}$$

yields a coupling between replicas. For a Gaussian distribution this involves

$$\left[\exp\left\{\beta \sum_{l\nu\alpha} H_{l\nu}\sigma_\nu^\alpha(l)\right\}\right]_{\text{av}} = \exp\left\{\frac{1}{2}\beta^2 \sum_{\substack{ij,\mu\nu\\ \alpha\beta}} [H_{i\mu}H_{j\nu}]_{\text{av}}\, \sigma_\mu^\alpha(i)\, \sigma_\nu^\beta(j)\right\} \tag{12}$$

which yields

$$\mathscr{H}_{\text{G}}^{(n,m)} = -\beta^{-1}K\left\{\sum_{\substack{ij,\mu\nu\\ \alpha\beta}} (v_{ij}\,\delta_{\alpha\beta}\,\delta_{\mu\nu} + K\varepsilon_1[H_{i\mu}H_{j\nu}]_{\text{av}})\, \sigma_\mu^\alpha(i)\, \sigma_\nu^\beta(j) + 2h\sum_{i\nu\alpha} \sigma_\nu^\alpha(i)\right\} \tag{13}$$

where $K = \beta J/2$ and $\varepsilon_1 = 2/J^2$, whereas for the non-Gaussian distribution of Eq. (8),

$$\mathscr{H}_{\text{non-G}}^{(n,m)} = -\beta^{-1}K\left\{\sum_{ij,\alpha\nu} v_{ij}\sigma_\nu^\alpha(i)\, \sigma_\nu^\alpha(j) + 2h\sum_{i\nu\alpha} \sigma_\nu^\alpha(i) + K^{-1}\sum_{i\nu} \ln\cosh\left[\beta h_0 \sum_\alpha \sigma_\nu^\alpha(i)\right]\right\} \tag{14}$$

This form includes, in contrast with Eq. (13), higher than quadratic terms in the spins, which have to be expanded in order to calculate the partition function. Nevertheless, the transformation to new Fourier-transformed variables[(4)] $\tilde{\sigma}_\nu^\alpha(p)$ such that $|\tilde{\sigma}_\nu^\alpha(p)|^2 = |\sigma_\nu^\alpha(p)|^2$, $\alpha = 1,\ldots, n$, with

$$\tilde{\sigma}_\nu^1(p) = m^{-1/2} \sum_\alpha \sigma_\nu^\alpha(p) \tag{15}$$

and that diagonalize the quadratic part of $\mathscr{H}^{(n,m)}$, for Gaussian and non-Gaussian distributions in the replica subspace, also serves to eliminate the expanded terms for the latter in the replica limit, as will be shown next.

We write the random-field average of the replicated partition function as

$$[Q_N^{(n,m)}(K,\varepsilon,h)]_{\rm av} = Z_N^{(n,m)}(K,\varepsilon,h)/Z_N^{(n,m)}(0,0,0) \tag{16}$$

where $h = H_0/J$, ε is a parameter that depends on the random-field distribution, and

$$Z_N^{(n,m)}(K,\varepsilon,h) = \int_{\tau - i\infty}^{\tau + i\infty} \prod_{i\alpha} \left(\frac{K}{2\pi i} dt_i^\alpha\right) \exp\left(Kn \sum_{i,\alpha} t_i^\alpha\right)$$
$$\times \int_{-\infty}^{\infty} \prod_{\substack{i\nu \\ \alpha}} [d\sigma_\nu^\alpha(i)]$$
$$\times \exp\left\{ -K \sum_{\substack{i\nu \\ \alpha}} t_i^\alpha [\sigma_\nu^\alpha(i)]^2 - \beta \mathscr{H}^{(n,m)}(\{\sigma_\nu^\alpha(j)\})\right\} \tag{17}$$

which includes the usual representation for the delta function, for each i and α,

$$\delta\left\{n - \sum_\nu [\sigma_\nu^\alpha(i)]^2\right\} = \frac{K}{2\pi i} \int_{-i\infty}^{i\infty} dt_i^\alpha \exp\left(K t_i^\alpha \left\{n - \sum_\nu [\sigma_\nu^\alpha(i)]^2\right\}\right) \tag{18}$$

that accounts for the n-vector model condition. In the limit $n \to \infty$ one may do a steepest descent integration in Eq. (17). Actually, the integration in the complex plane to calculate $[Q_N^{(n,m)}]_{\rm av}$ is not crucial and it can be avoided using Laplace's method, as in the case of the ordered n-vector model.[12]

Insertion of Eq. (14) and the assumption that the ln cosh there can *formally* be expanded in a power series yields

$$Z_N^{(n,m)} = \int_{\tau - i\infty}^{\tau + i\infty} \prod_{i\alpha} \left(\frac{K}{2\pi i} dt_i^\alpha\right) \exp\left[Kn \sum_{i,\alpha} t_i^\alpha + n \ln A_m(\{t_i\})\right] \tag{19}$$

where

$$A_m\{t_i\} = \int_{-\infty}^{\infty} \prod_{i\alpha} [d\sigma_n^\alpha(i)]$$
$$\times \exp\left(K\left\{\sum_{\substack{ij \\ \alpha}} v_{ij} \sigma_n^\alpha(i)\, \sigma_n^\alpha(j) + 2h \sum_{i\alpha} \sigma_n^\alpha(i)\right.\right.$$
$$\left.\left. - \sum_{i\alpha} t_i^\alpha [\sigma_n^\alpha(i)]^2 + K\varepsilon_2 \sum_i \left[\sum_\alpha \sigma_n^\alpha(i)\right]^2\right\}\right)$$
$$\times \left\{1 - \frac{1}{3} K^4 \varepsilon_2^2 \sum_i \left[\sum_\alpha \sigma_n^\alpha(i)\right]^4 + \cdots\right\} \tag{20}$$

for an arbitrary spin component taken here as $\nu = n$, $\varepsilon_2 = 2h_0^2/J^2$, and where the dots indicate higher order terms. Equation (15) yields

$$\sum_i \left[\sum_\alpha \sigma_\nu^\alpha(i) \right]^4 = m^2 \sum_i [\tilde{\sigma}_\nu^1(i)]^4 \tag{21}$$

and higher powers in m for higher order terms.

When the free energy per particle and degree of freedom,

$$-\beta f(\beta) = \lim_{N \to \infty} N^{-1} \lim_{m \to 0} \frac{d}{dm} \lim_{n \to \infty} n^{-1} \ln[Q_N^{(n,m)}]_{\rm av} \tag{22}$$

is calculated, the replica limit eliminates these terms and what remains is the effect of the quadratic part of $\mathscr{H}_{\text{non-}G}^{(n,m)}$, as in the case of a Gaussian distribution. This should be independent of the particular choice of a statistically independent non-Gaussian distribution and thus the same results should be obtained as for Gaussian distributions, in accordance with a comment by Hornreich and Schuster.[4] A crucial point here is the assumption of replica symmetry, implicit in Eq. (15) and in the existence of a single saddle-point parameter t_s for all replicas, satisfying the saddle-point equation

$$K + \lim_{N \to \infty} N^{-1} \lim_{m \to 0} m^{-1} \frac{d}{dt_s} \ln A_m(t_s) = 0 \tag{23}$$

We restrict the following discussion to Gaussian distributions of random fields.

Introducing Fourier lattice components of v_{ij},

$$v_{ij} = N^{-1} \sum_{\mathbf{k}} v(\mathbf{k}) \exp[-i(\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{k}] \tag{24}$$

with finite $v(0)$, one obtains

$$A_m(t_s) = \left(\frac{\pi}{K}\right)^{Nm/2} \left\{ \prod_{\mathbf{k}} [t_s - mK\varepsilon_1 \Delta L_\theta(k) - v(\mathbf{k})][t_s - v(\mathbf{k})]^{m-1} \right\}^{-1/2} \times \exp \frac{NmKh^2}{t_s - mK\varepsilon_1 \Delta L_\theta(0) - v(0)} \tag{25}$$

For a statistically independent distribution, where $L_\theta(0) = 1$, the order in which the thermodynamic and replica limits are taken at this point is irrelevant. Thus, as far as the free energy is concerned, one may write

$$\lim_{N \to \infty} \left[\frac{d}{dm} \phi_N^{(\rm SM)}(m)|_{m=0} \right] = \frac{d}{dm} [\lim_{N \to \infty} \phi_N^{(\rm SM)}(m)]|_{m=0} \tag{26}$$

for a statistically independent distribution of random fields, which leads to Eq. (5). In the case of long-range correlated fields, however, that order is crucial. Indeed, as pointed out above, $L_\theta(0)$ is finite only for finite N, leading to a well-defined nonzero argument in the exponential of Eq. (25). Note that if the thermodynamic limit is taken *before* the replica limit in the calculation of the free energy per particle, the argument of the exponential should vanish with $L_\theta(k)=k^{-|\theta|}\to\infty$ as $k\to 0$, and this yields the wrong result compared to the replica-independent calculation; see below.

The free energy and the saddle-point equation follow now as

$$\beta f(\beta)=\frac{1}{2}+\frac{1}{2}\ln 2K-Kt_s+\lim_{N\to\infty}(2N)^{-1}\sum_{\mathbf{k}}\left\{\ln[t_s-v(\mathbf{k})]\right.\\ \left.-K\varepsilon_1\,\Delta\,L_\theta(k)[t_s-v(\mathbf{k})]^{-1}\right\}-\frac{h^2K}{t_s-v(0)} \tag{27}$$

and

$$K=\lim_{N\to\infty}(2N)^{-1}\sum_{\mathbf{k}}\{[t_s-v(\mathbf{k})]^{-1}+K\varepsilon_1\,\Delta\,L_\theta(k)[t_s-v(\mathbf{k})]^{-2}\}\\ +\frac{h^2K}{[t_s-v(0)]^2} \tag{28}$$

which generalize the results of Hornreich and Schuster.[4] With $L_\theta(k)=k^{-|\theta|}$, for $\theta<0$, these equations yield in a standard way the dimensional shift of the thermodynamic properties for short-range interactions, with $v(k)=v(0)-v_0a^2k^2$ for small k, by $d\to d-2-|\theta|$, or for long-range interactions with $v(k)\approx v(0)-v_0a^\sigma k^\sigma$, $0<\sigma<2$, by $d\to d-\sigma-|\theta|$. These results agree with the recent ones of Ref. 7, obtained in the $n\to\infty$ limit of a generalization with random fields for the $1/n$ expansion of Ma.[17]

Similarly, other quantities, such as the Edwards–Anderson spin-glass order parameter,

$$q=\lim_{N\to\infty}N^{-1}\lim_{m\to 0}\lim_{n\to\infty}n^{-1}\sum_{i\nu}[\langle\sigma_\nu^\alpha(i)\rangle^2]_{\text{av}} \tag{29}$$

in which $\langle\cdots\rangle$ denotes the thermodynamic average, are easily obtained with the result

$$q=\mu^2+\lim_{N\to\infty}(2N)^{-1}\,\varepsilon_1\,\Delta\sum_{\mathbf{k}}\frac{L_\theta(k)}{[t_s-v(\mathbf{k})]^2} \tag{30}$$

where

$$\mu=h/[t_s-v(0)] \tag{31}$$

is the normalized magnetization per site. With long-range correlated random fields there is still a nonsingular q for all temperatures T, which vanishes only as $T \to \infty$, as in the case of uncorrelated Gaussian random fields discussed in Ref. 4.

3. THE SPHERICAL-MODEL LIMIT WITHOUT REPLICAS

The normalized partition function

$$Q_N^{(n)}(K, \{\tilde{h}_{iv}\}, h) = Z_N^{(n)}(K, \{\tilde{h}_{iv}\}, h)/Z_N^{(n)}(0, 0, 0) \tag{32}$$

for the n-vector model in a random field $\tilde{h}_{iv} = H_{iv}/J$ on site i can be obtained in standard way as for an ordered system, without resorting to the replica method, doing a steepest descent integration with $n \to \infty$ in

$$\begin{aligned} Z_N^{(n)}(K, \{\tilde{h}_{iv}\}, h) &= \int_{\tau - i\infty}^{\tau + i\infty} \prod_i \left(\frac{K}{2\pi i} dt_i\right) \exp\left(Kn \sum_i t_i\right) \\ &\quad \times \int_{-\infty}^{\infty} \prod_{iv} [d\sigma_v(i)] \exp\left\{-K \sum_{i,v} t_i [\sigma_v(i)]^2 - \beta H^{(n)}(\{\sigma_v(j)\})\right\} \end{aligned} \tag{33}$$

in which $H^{(n)}$ is the original Hamiltonian given by Eq. (7), in place of the effective Hamiltonian in Eq. (11). As far as the calculation of $Z_N^{(n)}$ is concerned, the random field acts as an additional ordering field. As usual in systems with quenched disorder, the random-field average is taken on the logarithm of $Q_N^{(n)}$, and this yields the free energy per site and degree of freedom as

$$-\beta f(\beta) = \lim_{N \to \infty} N^{-1} \lim_{n \to \infty} n^{-1} [\ln Q_N^{(n)}(K, \{\tilde{h}_{iv}\}, h)]_{\text{av}} \tag{34}$$

For $Z_N^{(n)}$ we obtain

$$Z_N^{(n)}(K, \{\tilde{h}_{iv}\}, h) = \exp\{n[KNt_s + \ln A(t_s)]\} \tag{35}$$

with

$$\begin{aligned} A(t_s) = \left(\frac{\pi}{K}\right)^{N/2} \exp\Bigg\{ &-\frac{1}{2} \sum_{\mathbf{k}} \ln[t_s - v(\mathbf{k})] + \frac{K}{2} \varepsilon_1 J^2 \sum_{\mathbf{k}} \frac{\tilde{h}(\mathbf{k})\,\tilde{h}(-\mathbf{k})}{t_s - v(\mathbf{k})} \\ &+ \frac{2KN^{1/2} h \tilde{h}(0)}{t_s - v(0)} + \frac{KNh^2}{t_s - v(0)} \Bigg\} \end{aligned} \tag{36}$$

in which $\tilde{h}(\mathbf{k})$ is the Fourier coefficient of $\tilde{h}_{iv}$, with $\tilde{h}(0) \equiv \tilde{h}(\mathbf{k})$ at $k = 0$, and t_s is given by

$$K + N^{-1} \frac{d}{dt_s} \ln A(t_s) = 0 \tag{37}$$

Taking now the average $[\ln A(t_s)]_{\text{av}}$ in both Eq. (35) *and* Eq. (37) yields the same expression as Eq. (27) for the free energy and

$$K + \lim_{N \to \infty} N^{-1} \frac{d}{dt_s} [\ln A(t_s)]_{\text{av}} = 0 \tag{38}$$

gives Eq. (28) for the saddle-point parameter. Note that for a statistically independent Gaussian distribution of random fields the result for the free energy agrees precisely with Refs. 2 and 6, taking into account the relationship between the spherical and mean-spherical models.[13]

This completes the replica-independent calculation, which enables a check on the order in which the limits have to be taken in the case of long-range correlated random fields, as discussed in the previous section.

4. CORRELATION FUNCTIONS

The work of the previous sections may be extended to the calculation of various correlation functions of interest, in particular the *net* correlation functions for spin component α,[18]

$$G_\alpha(\mathbf{r}_i, \mathbf{r}_j) = [\langle \sigma_\alpha(i)\, \sigma_\alpha(j) \rangle]_{\text{av}} - [\langle \sigma_\alpha(i) \rangle \langle \sigma_\alpha(j) \rangle]_{\text{av}} \tag{39}$$

$$C_\alpha(\mathbf{r}_i, \mathbf{r}_j) = [\langle \sigma_\alpha(i) \rangle \langle \sigma_\alpha(j) \rangle]_{\text{av}} - [\langle \sigma_\alpha(i) \rangle]_{\text{av}} [\langle \sigma_\alpha(j) \rangle]_{\text{av}} \tag{40}$$

and the correlation function $[\langle \sigma_\alpha(i)\, \sigma_\alpha(j) \rangle]_{\text{av}}$ that serves to study the first-order phase transition that appears below the ordering temperature T_c^0 of the nonrandom system when $\Delta \to 0$ for $d_c^0 < d < d_c$, in which d_c^0 and d_c are the lower critical dimensionalities for the pure and the random-field systems, respectively.

The calculation of Eqs. (39) and (40) is simplest without replicas, and introducing Fourier components

$$G_\alpha(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{N} \sum_{\mathbf{k}} G_\alpha(\mathbf{k}) \exp[-i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)] \tag{41}$$

$$C_\alpha(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{N} \sum_{\mathbf{k}} C_\alpha(\mathbf{k}) \exp[-i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)] \tag{42}$$

we find

$$G_\alpha(\mathbf{k}) = (2K)^{-1} [t_s - v(\mathbf{k})]^{-1} \tag{43}$$

and

$$C_\alpha(\mathbf{k}) = \frac{1}{2} \varepsilon_1 \Delta \frac{L_\theta(k)}{[t_s - v(\mathbf{k})]^2} \tag{44}$$

These make it possible to understand the saddle-point equation (28) in zero uniform field, which becomes

$$\lim_{N \to \infty} N^{-1} \sum_{\mathbf{k}} [G(\mathbf{k}) + C(\mathbf{k})] = 1 \tag{45}$$

a condition on the sum of the correlation functions at the origin in real-lattice space, generalizing a well-known result on spherical-like models for a nonrandom system.(13) Note that the right-hand side of Eq. (45) is *not* the free energy density as claimed in Ref. 7. On the other hand, we have

$$G_\alpha(k) = \Delta G_\alpha^2(k) \, |\mathbf{k}|^\theta \tag{46}$$

in agreement with Carra and Chalker.(7)

We turn now to our results for the correlation function(19) $[\langle \sigma_\alpha(i) \, \sigma_\alpha(j) \rangle]_{\text{av}}$, which is the quantity of interest for the study of the first-order phase transition in a random-field problem. It is possible to check by means of the spherical-model limit the low-temperature scaling assumptions for the relevant ordering fields h/T and Δ/T^2 that correspond to the magnetization and the Edwards–Anderson order parameter, respectively, as discussed by Aharony and Pytte,(16) but now for long-range correlated random fields.

For the Fourier components of $[\langle \sigma_\alpha(i) \, \sigma_\alpha(j) \rangle]_{\text{av}}$ we find, in the case of Gaussian random fields and short-range interactions taking a spherical Brillouin zone of radius π/a,[3]

$$S_{\alpha\alpha}(k, \kappa^{-1}) = \frac{\Delta}{J^2} \frac{k^\theta}{(\kappa^2 + k^2)^2} + \frac{1}{\beta J} \frac{1}{\kappa^2 + k^2} + \frac{1}{v_0 \pi^2} \delta_{k,0} h^2 \kappa^{-4} \tag{47}$$

Here $\kappa = \xi^{-1}$ is the inverse coherence or persistence length(19) that follows from the saddle-point equation in zero field, for fixed $T < T_c^0$, the critical temperature of the ordered system, with the result

$$\kappa = \left[\frac{f(d) \, \varepsilon_1}{1 - T/T_c^0} \right]^{1/(4 - \theta - d)} \Delta^{1/(4 - \theta - d)} \tag{48}$$

[3] See the text following Eq. (28).

in which

$$f(d)=(1/4)(C_d/v_0^2\pi^2)(d-2)\,\pi\,\csc[(d-2)\,\pi/2] \tag{49}$$

and $C_d=2^{-d}\pi^{(d-4)/2}\Gamma^{-1}(d/2)$. When $\theta<0$, these become the quantities appropriate for long-range correlated random fields. Note that $\xi\sim\Delta^{-\nu_\Delta}$, where now $\nu_\Delta=1/(4+|\theta|-d)$ in this case, consistent with a shifted lower critical dimensionality of $d_c=4+|\theta|$. Note also that in the spherical-model limit used here, these are relationships for *any* $T<T_c^0$ and generalize the low-temperature expectations of Aharony and Pytte to long-range correlated random fields. There is now a k dependence that corrects the Lorentzian squared when $\theta<0$ and, since $S_{\alpha\alpha}(k,\xi)$ is the structure factor, this new result may be of use experimentally.

Equation (47) also yields

$$S_{\alpha\alpha}(k,\xi)=\xi^d\bar{S}(k\xi,\,T\xi^{2-d},\,h\xi^2) \tag{50}$$

a general form in accordance with Ref. 16, for $\theta\geqslant 0$, in which the triple-scaling function in zero ordered field h is now given explicitly, for all x and y, by

$$\bar{S}(x,y,0)=A\left(\frac{d,T}{T_c^0}\right)\frac{x^\theta}{(1+x^2)^2}\left[1+\frac{k_B}{J}A^{-1}\left(d,\frac{T}{T_c^0}\right)(1+x^2)\,x^{-\theta}y\right] \tag{51}$$

where k_B is Boltzmann's constant and

$$A(d,T/T_c^0)\equiv[1/2f(d)](1-T/T_c^0)$$

These equations serve to check the general scaling assumption[16]

$$[\langle\sigma_\alpha(i)\,\sigma_\alpha(j)\rangle]_{av}=r_{ij}^{-(d-2+\eta_\Delta)}f(r_{ij}/\xi) \tag{52}$$

here also for long-range correlated random fields, where $r_{ij}\equiv|\mathbf{r}_i-\mathbf{r}_j|$ and $\eta_\Delta=2-d$.

5. SUMMARY OF RESULTS AND CONCLUDING REMARKS

We have shown that the replica limit may be interchanged with the limit $n\to\infty$ for the three kinds of random field distributions used in this work. In the case of statistically independent random fields, the thermodynamic and replica limits may also be interchanged, but this is not the case for long-range Gaussian correlated random fields in the presence of a nonzero ordering field. In all cases the correct results are obtained if the replica limit is taken *before* the thermodynamic limit. This is the right order of limits in the general case, except that one cannot perform explicit calculations with it. What saves us here from the need to interchange the thermodynamic and replica limits is the additional $n \to \infty$ limit.

In all we did here, with or without replicas, the $n \to \infty$ limit comes *before* the limit $N \to \infty$. Presumably these limits can be interchanged, but we have no proof for the moment. We remind the reader that already for an ordered system the hard part of the proof by Kac and Thompson[(12)] that justifies the interchange of limits is the limit $N \to \infty$ for fixed n, or both together N, $n \to \infty$.

We have also shown that the general scaling form for the structure factor of the n-vector model for finite n, discussed by Aharony and Pytte, also holds for long-range correlated random fields in the spherical-model limit. Although only the first two moments of the random-field distribution appear in this limit, the results should be quite more general, based on the belief that precisely these two moments should determine the behavior of the first-order phase transition below T_c^0 when $d_c^0 < d < d_c$.[4]

It is worth noting that the check with the structure factor of Ref. 16 indicates that the low-temperature scaling of the ordering fields h/T and Δ/T^2 with b^d, in which b is the length rescaling factor, based on the loop expansion in the nonlinear σ-model,[(21)] may be valid for all *fixed* $T < T_c^0$.

The results obtained in this work should be useful in carrying out a $1/n$ expansion with a random field.

[4] This point should be investigated further; the assumption here is that usually only the first two moments of the appropriate variable are relevant for the critical behavior of random systems. See Refs. 14, 15, and 20.

ACKNOWLEDGMENTS

This work was supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnològico (CNPq), Financiadora de Estudos a Projetos (FINEP) and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil.

REFERENCES

1. Y. Imry, *J. Stat. Phys.* **34**:849 (1984).
2. L. A. Pastur, *J. Stat. Phys.* **27**:119 (1982).
3. M. Schwartz, *Phys. Lett.* **76A**:408 (1980).
4. R. M. Hornreich and H. G. Schuster, *Phys. Rev. B* **26**:3929 (1982).
5. J. F. Perez and W. F. Wreszinski, *Rev. Bras. Fís.* **13**:477 (1983).
6. J. F. Perez, W. F. Wreszinski, and J. L. van Hemmen, *J. Stat. Phys.* **35**:89 (1984).
7. P. Carra and J. T. Chalker, *J. Phys. C* **18**:1919 (1985).

8. P. Lacour-Gayet and G. Toulouse, *J. Phys. (Paris)* **35**:425 (1974).
9. V. J. Emery, *Phys. Rev. B* **11**:239 (1975); S. F. Edwards and P. W. Anderson, *J. Phys. F* **5**:965 (1975).
10. J. L. van Hemmen and R. G. Palmer, *J. Phys. A* **12**:563 (1979).
11. H. E. Stanley, *Phys. Rev.* **176**:718 (1968).
12. M. Kac and C. J. Thompson, *Phys. Norv.* **5**:163 (1971).
13. G. S. Joyce, in *Phase Transition and Critical Phenomena*, Vol. 2, C. Domb and M. S. Green, eds. (Academic Press, New York, 1972).
14. M. Kardar, B. McClain, and C. Taylor, *Phys. Rev. B* **27**:5875 (1983).
15. M.-C. Chang and E. Abrahams, *Phys. Rev. B* **29**:201 (1984).
16. A. Aharony and E. Pytte, *Phys. Rev. B* **27**:5872 (1983).
17. S. K. Ma, in *Phase Transitions and Critical Phenomena*, Vol. 6, C. Domb and M. S. Green, eds. (Academic Press, New York, 1976).
18. G. Grinstein, *Phys. Rev. Lett.* **37**:944 (1976).
19. N. H. Berker and M. E. Fisher, *Phys. Rev. B* **26**:2507 (1982).
20. A. Weinrib and B. Halperin, *Phys. Rev. B* **27**:413 (1983).
21. E. Brézin and J. Zinn-Justin, *Phys. Rev. B* **14**:3110 (1976).